George J. Thaler

ON THE REALIZATION OF LINEAR MULTIVARIABLE CONTROL SYSTEMS.

UNITED STATES
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List of symbols and notations

Generally, upper case letters will be used to represent matrices, lower case letters to represent elements of matrices. However, for the sake of brevity, it is sometimes preferable to refer to matrices while speaking of their elements. For instance, instead of saying: "the elements of matrix (I-FT) have poles on the left half plane", the words "the elements of matrix" can be suppressed without causing ambiguity. Which elements of the said matrix are involved should be clear to the reader who follows the reasoning in a continuous fashion. The following symbols will be used in the sense as indicated:

P: plant transfer function matrix

G: open loop compensating filter transfer function matrix

C: cascade controller transfer function matrix

F: overall feedback controller transfer function matrix

H: inner feedback controller transfer function matrix

U: system input column matrix

X: plant input column matrix

Y: system output column matrix

E: error column matrix

T: overall transfer function matrix

I = identity matrix

s = Laplace operator

a, b, α, β ... = coefficients of polynomials in s

m = number of outputs of multivariable plant

n = number of inputs to multivariable plant

But m, n are also used as stated below:

m, n, v, w	= orders of polynomials in s
LHP	= left half of the s plane
RHP	= right half of the s plane

ON THE REALIZATION OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Chapter One

Multivariable control systems

I-1: Introduction:

The problem of synthesis of a linear multivariable control system consists in selecting a controller or controllers and a compensation scheme such that a set of specifications on transient and steady-state responses, disturbances and plant parameter variations be satisfied. Furthermore, the controller or controllers selected must be realizable in the physical sense.

A number of papers in the past have treated the synthesis problem and are mostly concerned with the satisfaction of specifications. Little attention has been paid to conditions concerning physical construction of the controllers. Recently R.J. Kavanagh[1] gave a detailed account on his study of realizability of non-interacting control starting from a given plant; but his problem was "mathematical" realizability, dealing with the existence or non-existence of a solution, and not with the physical construction of the controllers.

The purpose of this report is to present a method for recognizing whether or not, with a given plant and a given specification, the controller or controllers necessary to compensate the system can be built from physical components. It is also shown how a choice is possible among different compensation schemes to suit the problems of availability of components, reliability of components, cost, ruggedness, simplicity, or merely designer's taste.

I-2: Matrix representation of multivariable control systems:

A multivariable control system, just like a single variable system, is made up of a plant, or actuator, and one or several controllers, or compensators, interconnected in a certain fashion.

Each of these units will have several inputs and several outputs. It is then convenient to represent them with matrices, each element of which will be a transfer function relating an input to an output. The (i,j)th element relates the jth input to the ith output. Figure 1 represents a system with plant P and one controller C connected in cascade. $\underline{U}$, $\underline{X}$, $\underline{Y}$ are column matrices. $\underline{U}$ represents the system inputs; $\underline{X}$ the plant inputs, $\underline{Y}$ the system outputs, which are the same as the plant outputs.

Let n be the number of plant inputs and m the number of plant outputs, ie:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}$$

In practice, we always have $m \leqslant n$. It would make no sense to try to control more output variables than the number of input variables to which we have access. If $m = n$, P is a square matrix. If $m \leqslant n$, for convenience in the manipulation of matrices, we can add (n-m) artificial outputs, which we will monitor and feed back to the system

input. By so doing, we add extra rows to the plant matrix P, these extra rows having unity elements on the diagonal and zero elements elsewhere. As an example, a 3-output, 5-input plant matrix would become this square matrix:

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 2 shows what modification has been brought to our multivariable system. One can see that the number of system output is still 3. The number of system inputs is also 3 since no more than 3 different inputs are needed to command 3 system outputs.

As a result of this artificial manipulation, plant matrices and controller matrices can always be made square. Then only square matrices shall be dealt with.

I-3: Interaction and non-interaction:

There have been several attempts to define what is meant by non-interaction in a multivariable system[1, 2, 3]. The simplest but also most logical one seems to be the following: non-interaction is said to occur when there is a one-to-one relationship between system inputs and system outputs. In other words each system input controls one definite output and that one only. Conversely, whenever one input not only controls its own output but also changes one or more other outputs, the system is said to be interacting.

Ideally, non-interaction is the goal in the design of multivariable systems. But from an engineering viewpoint, this is not always physically realizable, nor is it always desirable, for a number of reasons, practical, economical, commercial, among others.

It is then reasonable to proceed as follows: attempt to design the system for non-interaction, and build it if it is judged satisfactory for the various reasons mentioned above. If non-interaction is unrealizable or impractical, then go to the next-to-best solution, which is a system with some minimum amount of interaction, or with a specified amount of interaction as the case may be, until a compromise is reached between satisfactory performance and physical and economical realization.

Chapter Two

Compensation design

II-1: Selection of a compensation scheme:

In the current literature, a current practice has been to arbitrarily select a compensation scheme and solve the problem for that particular scheme. Some writers assume one cascade and one feedback controller[4], others restrict their treatment to a system with only a cascade controller[5, 6]. Another assumes a feedback controller only, and goes as far as restricting himself to a diagonal plant matrix[7].

It is true that some compensation schemes are more commonly used than others, but it is also true that some other scheme may serve a particular purpose better, or suit existing conditions better in a given situation. Consequently it is preferable for the designer to gain an insight to several compensation schemes and try to select the most suitable.

Figure 3 shows 3 possible structures using both cascade and feedback controllers.

While schemes no 1 and 3 seem to be more general and reasonable, in some cases the designer may have to use scheme no 2, for example when end-to-end feedback for the overall system is impractical or impossible due to geographical conditions.

II-2: Filter design:

In order to avoid restricting the design to any particular arrangement, it is proposed that synthesis be based on the simplest

system configuration as indicated on Figure 4, where G represents a multivariable cascade filter. No feedback of any kind is involved.

For a given desired overall response, ie: a specified overall transfer function, say T, we have:

$$PG = T$$

$$\text{or} \qquad \boxed{G = P^{-1}T} \tag{1}$$

There is no difficulty in obtaining the expression of G. The operations in (1) are matrix operations, hence order is important since generally $PG \neq GP$.

Note that, as stated before, design was started off by assuming non-interaction, therefore T, the overall transfer matrix is diagonal, and each diagonal element t_{ii} is the desired transfer function of the ith variable:

$$\begin{bmatrix} y_1 \\ y_2 \\ \circ \\ \circ \\ \circ \\ y_n \end{bmatrix} = \begin{bmatrix} t_{11} & 0 & 0 & 0 \\ 0 & t_{22} & 0 & 0 \\ - & - & - & - \\ - & - & - & t_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \circ \\ \circ \\ \circ \\ u_n \end{bmatrix} \tag{2}$$

In the next step, filter G is split into cascade controller C and/or feedback controller(s) F and H. Several possibilities then occur

- if none of the C, F, H controllers thus obtained are physically or economically realizable, it can be concluded that non-interaction is not possible. One must then go to the next best solution, which

is the scope of another report to follow

- if only one of the above schemes is realizable, there is no choice but to take it
- if more than one schemes are realizable, the designer has the privilege of choosimg one of them in the light of commercial and economical conditions, availability of components, reliability, ruggedness or his own personal taste.

The design based on this GP configuration has 2 further advantages:

- in cases where it is necessary to design the system with interaction, it is easier to see which element is to be redesigned in order to make one particular interaction, say the (i, j)th, meet the limits specified
- as will be shown later in this report, examination of G alone is sufficient to determine whether or not the compensating schemes of Figure 3 are physically realizable. This permits the designer to avoid wasting further time on an impossibility.

In the compensation design using filter G, it is convenient to have a set of formulas for going from G to C, F and/or H.

For scheme no 1 -

Specifications are given in the form of an overall transfer matrix T, or may be converted into such a matrix. The writer hopes to undertake a more detailed study of this matter in the next paper. By definition of T:

$$Y = TU \tag{3}$$

But from figure:

$$Y = PX \tag{4}$$

$$X = CE \tag{5}$$

$$E = U - FY \tag{6}$$

Again the order of operations is important since matrix quantities are involved.

From (4), (5) and (6): $Y = PC(U-FY)$

or : $[I + PCF]\, Y = PCU$ (7)

where I is (nxn) identity matrix.

Premultication of both sides of (7) by $[I + PCF]^{-1}$ yields:

$$Y = [I + PCF]^{-1}PCU \tag{8}$$

Comparison of equations (3) and (8) yields:

$$T = [I + PCF]^{-1}PC \tag{9}$$

But we know that $T = PG$ (10)

Equating (9) and (10) and premultiplying both sides by $[I + PCF]$:

$$PC = [I + PCF]\; PG$$

or $PC = PG + PCFPG$

Rearranging $\cancel{P}C\,[I - FPG] = \cancel{P}G$

Noting that $PG = T$, $\boxed{C = G\,[I - FT]^{-1}}$ (11)

(11) is the relation between G on one hand and C, F on the other. This is one equation with two unknowns, C and F. The designer is free to choose one, say F, the other variable, C,

is then determined. Needless to say, F, that is the elements of F, must be chosen so that they are physically realizable. It would be helpful if knowledge of G alone can determine whether or not a C, F configuration can be realized. This is the purpose of the last part of the paper.

For scheme no 2 -

$$T = PG$$

Calculations similar to the previous case lead to:

$$T = [I + PH]^{-1}PC$$

Equating the 2 equations : $PG = [I + PH]^{-1}PC$

Premultiplying by $[I - PH]$: $PC = PG + PHPG$

Since $PG = T$, $$C = G + HT \qquad (12)$$

Equation (12) relates G with C and H. As before, selection of H permits calculation of C or vice versa.

Note that, in scheme no 1, a possible choice is to take $F = I$, ie: unity feedback without crossfeed. This is a very realistic choice since in a control system it is desirable to compare each input with its own output on a one-to-one basis. But here in scheme no 2, feedback matrix H will generally be non-diagonal since there is no one-to-one correspondence between y's and x's.

For scheme no 3 -

$$T = PG$$

Successive reduction of block diagram leads to:

$$T = PG = [I + (I + PH)^{-1}PCF]^{-1} (I + PH)^{-1}PC$$

Premultiplication by $[I + (I + PH)^{-1}PCF]$ yields:

$$PG + (I + PH)^{-1}PCFPG = (I + PH)^{-1}PC$$

Rearranging:

$$PG = (I + PH)^{-1}PC [I - FPG]$$

Premultiply by $(I + PH)$:

$$(I + PH) PG = PC (I - FPG)$$

Postmultiply by $(I - FPG)^{-1}$:

$$(I + PH) PG (I - FPG)^{-1} = PC$$

or: $PC = (PG + PHPG) (I - FPG)^{-1}$

or: $\cancel{P}C = \cancel{P} (G + HPG) (I - FPG)^{-1}$

ie $\boxed{C = (G + HT) (I - FT)^{-1}}$ (13)

Equation (13) relates G with C, F and H. In the most general case of unity feedback, F = I and (13) becomes

$$C = (G + HT)\,(I - T)^{-1} \tag{14}$$

Selection of one of the 2 unknowns, C and H, determines the other. This is the essence of the designer's task.

Note that (13) is a more general expression of (11). Letting H = 0 in (13), one obtains (11).

The following example shows how by use of equations (11) (12) (13), a sensible choice between the above compensating schemes is possible.

Example - Given the plant to be controlled

$$P = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+20} \\ \frac{1}{s+20} & \frac{1}{s+3} \end{pmatrix}$$

It is desired that the transient responses y_1, y_2 have no overshoot and have a short rise time as expressed by the overall transfer functions $t_{11} = t_{22} = \frac{4}{(s+2)^2}$

First obtain the transfer matrix for filter G:

$$G = P^{-1}T$$

$$P^{-1} = \frac{1}{\Delta_p}\begin{bmatrix} \frac{1}{s+3} & -\frac{1}{s+20} \\ -\frac{1}{s+20} & \frac{1}{s+1} \end{bmatrix} = \frac{1}{\frac{1}{(s+1)\,(s+3)} - \frac{1}{(s+20)^2}}\begin{bmatrix} \frac{1}{s+3} & -\frac{1}{s+20} \\ -\frac{1}{s+20} & \frac{1}{s+1} \end{bmatrix}$$

$$P^{-1} = \frac{(s+1)\ (s+3)\ (s+20)^2}{(s+20)^2 - (s+1)\ (s+3)} \begin{bmatrix} \frac{1}{s+3} & -\frac{1}{s+20} \\ -\frac{1}{s+20} & \frac{1}{s+1} \end{bmatrix} =$$

$$= \frac{(s+1)\ (s+3)\ (s+20)^2}{36\ (s+11)} \begin{bmatrix} \frac{1}{s+3} & -\frac{1}{s+20} \\ -\frac{1}{s+20} & \frac{1}{s+1} \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{(s+1)\ (s+20)^2}{36\ (s+11)} & -\frac{(s+1)\ (s+3)\ (s+20)}{36\ (s+11)} \\ -\frac{(s+1)\ (s+3)\ (s+20)}{36\ (s+11)} & \frac{(s+3)\ (s+20)^2}{36\ (s+11)} \end{bmatrix}$$

$$G = P^{-1}T = \begin{bmatrix} \frac{(s+1)\ (s+20)^2}{36\ (s+11)} & -\frac{(s+1)\ (s+3)\ (s+20)}{36\ (s+11)} \\ -\frac{(s+1)\ (s+3)\ (s+20)}{36\ (s+11)} & \frac{(s+3)\ (s+20)^2}{36\ (s+11)} \end{bmatrix} \begin{bmatrix} \frac{4}{(s+2)^2} & 0 \\ 0 & \frac{4}{(s+2)^2} \end{bmatrix}$$

$$G = \begin{bmatrix} \frac{1}{9} \frac{(s+1)\ (s+20)^2}{(s+2)^2\ (s+11)} & -\frac{1}{9} \frac{(s+1)\ (s+3)\ (s+20)}{(s+2)^2\ (s+11)} \\ -\frac{1}{9} \frac{(s+1)\ (s+3)\ (s+20)}{(s+2)^2\ (s+11)} & \frac{1}{9} \frac{(s+3)\ (s+20)^2}{(s+2)^2\ (s+11)} \end{bmatrix} \qquad (15)$$

The above 4 elements of filter G, g_{11} g_{12} g_{21} and g_{22} are all physically realizable (see Chapter Three). Theoretically, such a filter when connected in cascade with plant P, would give the desired non-interacting responses. However, it is

well known that some feedback is desirable and even necessary, to minimize the effect of unwanted disturbances introduced anywhere between system inputs and outputs. Feedback is also necessary to reduce the sensitivity of overall system to plant parameter changes. Consequently, G is now converted into C, F, H as indicated in the various compensation schemes of Figure 3.

Scheme no 1 -

From equation (11):

$$C = G(I - FT)^{-1}$$

As discussed before, this is one equation for two unknowns, C, and F. The designer's job is to select one (and compute the other) in the way that best suits his particular problem. A possible choice here would be to take F = I, I being the (nxn) identity matrix. This corresponds to unity feedback from each system output to the corresponding system input. This is not the only possible value for F but is taken for this illustrative example.

(11) then becomes $$C = G(I - T)^{-1} \tag{16}$$

But $(I - T) = \begin{pmatrix} 1 - \dfrac{4}{(s+2)^2} & 0 \\ 0 & 1 - \dfrac{4}{(s+2)^2} \end{pmatrix} = \begin{pmatrix} \dfrac{s(s+4)}{(s+2)^2} & 0 \\ 0 & \dfrac{s(s+4)}{(s+2)^2} \end{pmatrix}$

$$(I - T)^{-1} = \begin{pmatrix} \frac{(s+2)^2}{s(s+4)} & 0 \\ 0 & \frac{(s+2)^2}{s(s+4)} \end{pmatrix} \tag{17}$$

Use of expressions (15) and (17) into equation (16) yields the expression for C:

$$C = \frac{1}{9} \frac{(s+20)}{s(s+4)\ (s+11)} \begin{bmatrix} (s+1)\ (s+20) & -\ (s+1)\ (s+3) \\ -\ (s+1)\ (s+3) & (s+3)\ (s+20) \end{bmatrix} \tag{18}$$

The elements of C are realizable as passive networks. Comparison of respective elements of (15) and (18) shows that C is no more complicated than G as far as physical construction is concerned. Considering the advantages of feedback, scheme no 1 is therefore acceptable.

Scheme no 2 -

From eq (12):

$$C = G + HT$$

In this case H is non-diagonal generally ie:

$$HT = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} t_{11} & 0 \\ 0 & t_{22} \end{pmatrix} = \begin{pmatrix} h_{11}t_{11} & h_{12}t_{22} \\ h_{21}t_{11} & h_{22}t_{22} \end{pmatrix}$$

$$C = \begin{bmatrix} g_{11} + h_{11}t_{11} & g_{12} + h_{12}t_{22} \\ g_{21} + h_{21}t_{11} & g_{22} + h_{22}t_{22} \end{bmatrix}$$

Here again, selection of H yields the expression of C. As an example, examine the element C_{11}, assuming $h_{11} = 1$. This is not a reasonable assumption, since, unlike scheme no 1, here H does not represent an end-to-end feedback comparing reference with actual output. However this oversimplified assumption may help the designer reach an insight to other possible choices.

Thus, assuming $h_{11} = 1$, and taking g_{11} from eq (15), we have:

$$c_{11} = \frac{1}{9}\,\frac{(s+1)\,(s+20)^2}{(s+2)^2\,(s+11)} + \frac{4}{(s+2)^2} = \frac{1}{9}\,\frac{s^3 + 41s^2 + 476s + 796}{(s+2)^2\,(s+11)}$$

This element c_{11} is physically realizable and is no more complicated to build than the elements of C in scheme no 1 (equation 18).

However it has been assumed $h_{11} = 1$. If it is necessary to realize a certain loop transfer function L = PH, in order to satisfy specifications on sensitivity to parameter variations (see loop shaping, chaps 3 and 10 reference 8), h_{11} is more likely to have the form $\frac{k(s+\alpha)}{(s+\beta)}$. Then, c_{11} will be, at best, more complicated than the corresponding cascade controller elements of scheme no 1 (more poles, more zeros to realize). In addition, it costs more to build both H and C versus C alone.

Despite the higher cost, this scheme has to be adopted whenever an overall feedback is difficult or impossible.

Scheme no 3 -

Assuming overall unity feedback (F = I), eq (14) gives:

$$C = (G + HT)\,(I - T)^{-1}$$

Selection of H yields a computed expression for C.

$$C = \begin{bmatrix} g_{11} + h_{11}t_{11} & g_{12} + h_{12}t_{22} \\ g_{21} + h_{21}t_{11} & g_{22} + h_{22}t_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{1 - t_{11}} & 0 \\ 0 & \frac{1}{1 - t_{22}} \end{bmatrix}$$

Taking the first element c_{11} as an example:

$$c_{11} = (g_{11} + h_{11}t_{11}) \frac{1}{1 - t_{11}} = \frac{g_{11}}{1 - t_{11}} + h_{11} \frac{t_{11}}{1 - t_{11}}$$

The first term on the right hand side has been obtained from scheme no 1 (eq 18). Thus:

$$c_{11} = \frac{1}{9} \frac{(s+1)(s+20)^2}{s(s+4)(s+11)} + h_{11} \frac{4}{s(s+4)} \tag{19}$$

(19) shows that

1 - if $h_{11} = 0$, one goes back to scheme no 1.

2 - if one takes the oversimplifying but not realistic assumption $h_{11} = 1$, c_{11} is physically realizable and not more complicated than c_{11} of scheme no 1.

3 - if, as is usually the case, h_{11} must assume the form $k \frac{s+\alpha}{s+\beta}$ to satisfy a number of requirements, sensitivity and others, then c_{11} becomes, at best, more complicated than c_{11} of equation (18), as far as physical realization is concerned.

Conclusion:

For this example it may be concluded that scheme no 1 is simplest and most economical to build, unless difficulty of realizing overall feedback leads to scheme no 2 in which case more hardware will be needed in the controller H; or unless special problems concerning

loop shaping leads to scheme no 3, where two controllers need be built in lieu of one. Other problems arise where construction of H and C as in scheme no 3 is simpler than that of C alone as in scheme no 1. Still other problems arise where one scheme leads to controllers with real poles, while another scheme leads to controllers with complex poles realizable only with active networks which the designer may or may not wish to use.

Chapter Three

Physical realizability

For a rational function of the Laplace transform variable s to be physically realizable as a voltage transfer function, three conditions must be met[9]:

Condition 1 - The coefficients of the function must be real

Condition 2 - The order of the numerator must be less than or equal to the order of the denominator

Condition 3 - There can be no poles in the right-half plane, and poles on the imaginary axis must be simple

It is important to note that the above conditions are in connection with unilateral networks. A network giving $\left.\frac{V_2}{V_1}\right|_{I_2=0} = F(s)$ does not give $\left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{1}{F(s)}$

In the following, it will be seen whether any relationship exists between physical realizability of G on one hand, and that of C, H, F on the other.

Scheme no 1 -

Since comparison between each reference with its own input is generally desired, F will be made diagonal. Further, elements of F must be realizable functions. Also, the elements of T, as specified, are realizable transfer functions. For non-interaction, T is diagonal and the elements of FT have the form $f_{ii}t_{ii}$ where $i = 1, 2, \ldots\ldots n$.

Both f_{ii} and t_{ii} are realizable. Is their product realizable also? Condition 1 is satisfied since the product of real

coefficients yields real coefficients. Condition 2 is satisfied

since: order of numerator f_{ii} $\leqslant$ order of denominator f_{ii}

order of numerator t_{ii} $\leqslant$ order of denominator t_{ii}

by addition: order of numerator $f_{ii}t_{ii}$ $\leqslant$ order of denominator $f_{ii}t_{ii}$

Finally, condition 3 is also satisfied since the poles of $f_{ii}t_{ii}$ are also poles of f_{ii} or t_{ii}. Consequently, $f_{ii}t_{ii}$, ie: the elements of FT, are realizable transfer functions.

From the realizability property of the elements of FT the 2 following consequences may be derived:

a) <u>All coefficients of $(I - FT)^{-1}$ are real.</u> Proof:

Since FT is diagonal, (I - FT) is also diagonal, and elements of $(I - FT)^{-1}$ are just inverses of elements of (I - FT).

Since $f_{ii}t_{ii}$ is realizable, it may be written as:

$$f_{ii}t_{ii} = \frac{a_n s^n + \ldots. + a_1 s + a_o}{b_m s^m + \ldots. + b_1 s + b_o}$$

where $a_n \ldots a_o$, $b_m \ldots b_o$ are real.

$$1 - f_{ii}t_{ii} = 1 - \frac{a_n s^n + \ldots. + a_o}{b_m s^m + \ldots. + b_o}$$

The coefficients of the denominator of $1 - f_{ii}t_{ii}$ are the same as those of the denominator of $f_{ii}t_{ii}$. The coefficients of the numerator of $(1 - f_{ii}t_{ii})$ are differences of real numbers ie: real numbers themselves. Thus, all coefficients of the elements of (I - FT) are real. If these elements are inverted,

the elements of $(I - FT)^{-1}$ are obtained, whose coefficients are the same, ie: still real numbers. qed.

b) <u>Generally, order of numerator of $(I - FT)^{-1}$ equals order of denominator of $(I - FT)^{-1}$.</u> The reason for using the word "generally" will be mentioned below.

For f_{ii} to be a realizable function, it must satisfy the condition order of numerator $\leq$ order of denominator. t_{ii}, the desired overall response function, <u>generally</u> satisfies the condition order of numerator < order of denominator[10]. Adding the two inequalities together, one obtains:

order of numerator $f_{ii}t_{ii}$ < order of denominator $f_{ii}t_{ii}$.

Note that this is a < sign, and no longer the $\leq$ sign.

Then: order of numerator $(1 - f_{ii}t_{ii})$ =

= order of denominator $(1 - f_{ii}t_{ii})$

The < sign becomes = sign due to the addition of 1 to the element.

Also: order of numerator $(1 - f_{ii}t_{ii})^{-1}$ =

= order of denominator $(1 - f_{ii}t_{ii})^{-1}$

Thus, statement is proved.

Now go back to the original problem: a criterion for realizability of scheme no 1.

$$C = G(I - FT)^{-1}$$

For C to be realizable:

1 - all coefficients of C must be real. This amounts to saying that all coefficients of G must be real, since those of

$(I - FT)^{-1}$ are real (from a above), and product of real numbers gives real numbers.

2 - order of numerator of C must be $\leqslant$ order of denominator of C. This obviously is equivalent to the condition: order of numerator of G $\leqslant$ order of denominator of G, since from b above, order of numerator of $(I - FT)^{-1}$ =

= order of denominator of $(I - FT)^{-1}$.

3 - poles of C must be on LHP and j-axis poles must be simple. First note that, since $C = G(I - FT)^{-1}$, poles of C are either poles of G, or poles of $(I - FT)^{-1}$. If G has any forbidden poles, one may ask the question whether it is possible that they be cancelled by similar zeros of $(I - FT)^{-1}$ ie: poles of (I - FT). The answer is no, since poles of (I - FT) are the same as poles of FT, and therefore cannot be forbidden ones. Thus, one condition for C to have acceptable poles is that G itself have acceptable poles. The other condition for C to have acceptable poles, obviously, is that poles of $(I - FT)^{-1}$ be on the LHP, or be simple on the j-axis. Note that the elements of $(I - FT)^{-1}$ are merely the inverses of those of (I - FT) since this matrix is diagonal.

In summary, it has been determined that for the scheme no 1 to be realizable, G must satisfy all 3 conditions, ie: be realizable itself; and in addition, poles of $(I - FT)^{-1}$ be on the LHP, or be simple on the j-axis. With a given T, this last condition serves as a guide in the selection of a proper expression for F.

As an example, if desired t_{11} is $\frac{4}{(s+2)^2}$, $f_{11} = 1$ leads to a realizable feedback scheme since $1 - f_{11}t_{11} = \frac{s(s+4)}{(s+2)^2}$,

$(1 - f_{11}t_{11})^{-1} = \frac{(s+2)^2}{s(s+4)}$, whose poles are acceptable.

The value $f_{11} = 3$ leads to an unrealizable feedback scheme since

$$1 - f_{11}t_{11} = 1 - \frac{12}{(s+2)^2} = \frac{s^2+4s-8}{(s+2)^2} = \frac{(s+5.46)\ (s-1.46)}{(s+2)^2}$$

$(1 - f_{11}t_{11})^{-1} = \frac{(s+2)^2}{(s+5.46)\ (s-1.46)}$ whose pole at + 1.46 is unacceptable.

Scheme no 2 -

It has been said previously that, while in scheme no 1, feedback controller matrix F is generally diagonal, here in scheme no 2, there is no reason to assume H diagonal since H is not intended to compare each overall input with its own output. Therefore, the general expression for C, from eq (12), is:

$$\begin{bmatrix} C_{11} \cdots C_{1n} \\ \vdots \\ C_{n1} \cdots C_{nn} \end{bmatrix} = \begin{bmatrix} g_{11} \cdots g_{1n} \\ \vdots \\ g_{n1} \cdots g_{nn} \end{bmatrix} + \begin{bmatrix} h_{11} \cdots h_{1n} \\ \vdots \\ h_{n1} \cdots h_{nn} \end{bmatrix} \begin{bmatrix} t_{11} & & 0 \\ & \ddots & \\ 0 & & t_{nn} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} \cdots C_{1n} \\ \vdots \quad \vdots \\ C_{n1} \cdots C_{nn} \end{bmatrix} = \begin{bmatrix} (g_{11} + h_{11}t_{11}) \cdots\cdots (g_{1n} + h_{1n}t_{nn}) \\ (g_{21} + h_{21}t_{11}) \cdots\cdots (g_{2n} + h_{2n}t_{nn}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (g_{n1} + h_{n1}t_{11}) \cdots\cdots (g_{nn} + h_{nn}t_{nn}) \end{bmatrix}$$

The general expression for C_{ij} may be written as:

$$C_{ij} = g_{ij} + h_{ij}t_{jj}$$

The element t_{jj} is realizable, and h_{ij} has to be chosen realizable, and consequently $h_{ij}t_{jj}$ is realizable, ie:

$$h_{ij}t_{jj} = \frac{\alpha_v s^v + \dots + \alpha_1 s + \alpha_o}{\beta_w s^w + \dots + \beta_1 s + \beta_o}$$

where $\alpha_v \dots \alpha_o$, $\beta_w \dots \beta_o$ are real coefficients, $v \leqslant w$; and all poles are on LHP, and simple if they are on the j-axis.

Let $g_{ij} = \dfrac{a_n s^n + \dots + a_1 s + a_o}{b_m s^m + \dots + b_1 s + b_o}$. Then;

$C_{ij} = g_{ij} + h_{ij}t_{jj} =$

$$= \frac{\left(a_n s^n + \dots + a_o\right)\left(\beta_w s^w + \dots + \beta_o\right) + \left(\alpha_v s^v + \dots + \alpha_o\right)\left(b_m s^m + \dots + b_o\right)}{\left(b_m s^m + \dots + b_o\right)\left(\beta_w s^w + \dots + \beta_o\right)} \tag{20}$$

What are the conditions for C_{ij} to be realizable?

1 - all coefficients of C_{ij} must be real: For the denominator, since $\beta_w \ldots \beta_o$ are real, $b_m \ldots b_o$ must be real also.

For the numerator, coefficients are made up by additions such as $(a_n\beta_w + \alpha_v b_m)$. It was assumed that α_v, β_w are real. It was also determined that b_m must be real. The only unknown left is a_n. For the sum $(a_n\beta_w + \alpha_v b_m)$ to be real, a_n also has to be real.

Hence, this first condition is equivalent to saying that all coefficients of g_{ij} must be real.

2 - order of numerator of C_{ij} must be $\leqslant$ order of denominator.

Since $h_{ij}t_{jj}$ is realizable, $v \leqslant w$. Looking at expression of C_{ij} in (20) and focusing one's attention on the superscripts, it is seen that 2 cases may happen: either $nw \geqslant mv$ or $nw \leqslant mv$. If $nw \geqslant mv$, the order of numerator of C_{ij} will be nw and condition 2 becomes $nw \leqslant mw$, ie: $\underline{n \leqslant m}$. If $nw \leqslant mv$, order of numerator of C_{ij} will be mv and condition 2 becomes $mv \leqslant mw$, ie: $v \leqslant w$ which is known to be true.

Thus, condition 2 is equivalent to saying that $n \leqslant m$, ie: order of numerator of g_{ij} must be $\leqslant$ order of denominator of g_{ij}.

3 - poles of C_{ij} must be on LHP, and must be simple if they are on the j-axis.

Eq (20) shows that poles of C_{ij} are merely poles of g_{ij} and poles of $h_{ij}t_{jj}$. Those of $h_{ij}t_{jj}$ are acceptable since $h_{ij}t_{jj}$ is realizable. Then, to satisfy condition 3, poles

poles of g_{ij} must be acceptable also, unless there are some RHP zeros of numerator of (20) to cancel the forbidden poles of g_{ij}. But the following reasoning shows that this "unless" cannot happen. Refer to numerator of (20). Suppose g_{ij} has a forbidden pole, say at +3, ie: $\left(b_m s^m + \ldots . + b_o\right)$ contains the factor (s-3). Then, the second term of numerator of (20) contains (s-3) as a factor also. For a cancellation to be possible, the first term of numerator of (20) must also contain factor (s-3). This is impossible since $\left(\beta_w s^w + \ldots . \beta_o\right)$ is denominator of a realizable function and cannot have forbidden roots, and $\left(a_n s^n + \ldots . + a_o\right)$ cannot contain (s-3) either (it would have cancelled with (s-3) of denominator of g_{ij} and there would have been no problem).
Therefore, condition 3 is equivalent to saying that the poles of g_{ij} itself must be acceptable.
To summarize: the condition for feedback scheme no 2 to be realizable is that G itself be realizable.

Scheme no 3 -

$$C = (G + HT)\,(I - FT)^{-1}$$

It is true that scheme no 3 is the most general scheme and includes the 2 others as particular cases. If H is taken = 0, scheme no 1 is obtained. If F = 0, scheme no 2 is obtained. However it is easier to treat these 2 schemes first, and extend the results to the more general case of scheme no 3.

By comparison with scheme no 1, it is clearly seen that what

applied for G, now becomes applicable to (G + HT). Thus one may use the results for scheme no 1 on page 21 for the present case.

From page 21, derive the following:

For scheme no 3 to be realizable, one must have:

a - (G + HT) realizable

b - poles of $(I - FT)^{-1}$ on LHP, or simple if on the j-axis.

For (a), one may use the results of scheme no 2 on top of page 25, which say that for (G + HT) to be realizable, G itself must be realizable. The following conclusion is arrived at: <u>Realizability of scheme no 3 is assured if</u>

<u>a - G itself is realizable</u>

<u>b - poles of $(I - FT)^{-1}$ are on LHP, or simple on j-axis.</u>

This is the same conclusion as for scheme no 1. Hence, whenever scheme no 1 can be built, so can scheme no 3.

<u>III-4: Generalization of results to interacting systems:</u>

In the above sections, T was taken as a diagonal matrix, ie: the system was designed for non-interacting response. Under such condition, it was found that realizability of all 3 schemes is assured if the following 2 conditions are fulfilled:

a - the elements of filter G are physically realizable

b - the poles of the elements of $(I - FT)^{-1}$ are on the LHP, or are simple if they are on the imaginary axis

For schemes no 1 and 3 where $F \neq 0$, condition b guides the designer in the choice of an expression for F. In other words, the designer has one equation, to solve for 2 unknowns, but the choice of one of the 2 unknowns is not quite a free choice. For

scheme no 2, where $F = 0$, condition b becomes trivial and may be suppressed.

It is desired now to extend the above result to the more general case of interacting systems, in which T is a non-diagonal matrix.

It turns out, after a somewhat tedious proof, that the above is still applicable to the case of non-diagonal T matrix.

In the above sections, where T is diagonal, $(I - FT)$ is also diagonal (since F is diagonal for one-to-one feedback). Thus the elements of $(I - FT)^{-1}$ are merely the inverses of the diagonal elements of $(I - FT)$. In other words, the poles of the elements of $(I - FT)^{-1}$ are the zeros of corresponding elements of $(I - FT)$.

For the more general case of a non-diagonal T matrix, we have

$$(I - FT)^{-1} = \frac{1}{|I - FT|}\left[\mathrm{adj}(I - FT)\right]$$

where $|I - FT|$ is the determinant of the matrix $(I - FT)$. Then the poles of the elements of $(I - FT)^{-1}$ are roots of the equation

$$\left|I - FT \quad = 0\right|$$

plus some others coming from the expression of adj $(I - FT)$.

The remainder of this section is devoted to the proof as applied to the general case, and the result stated at the end of the section.

x^x x

Start with:

$$C = (G + HT)\ (I - FT)^{-1}$$

For convenience in notation, let $(I - FT)^{-1}$ be defined as L matrix

$$(I - FT)^{-1} \triangleq L$$

three statements will be made before the final proof is undertaken.

<u>statement 1:</u> elements of L have all real coefficients

<u>statement 2:</u> order of numerator of elements of L $\leq$ order of denominator

<u>statement 3:</u> if 2 transfer functions are physically realizable, their sum is a realizable transfer function.

Note that the first two statements are similar to the 2 statements on pages 19 and 20 concerning the diagonal-T case.

Statement no 3 is proved in Appendix 1.

An illustration using (2x2) matrices clearly shows the validity of statements 1 and 2 for the general case with non-diagonal T matrix.

$$L^{-1} \triangleq I - FT = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

$$L^{-1} = \begin{pmatrix} 1 - f_{11}t_{11} & - f_{11}t_{12} \\ - f_{22}t_{21} & 1 - f_{22}t_{22} \end{pmatrix}$$

$$L = \frac{1}{L^{-1}} \left[\text{adj } L^{-1}\right] \tag{21}$$

$L^{-1} =$

$$= \frac{1}{(1 - f_{11}t_{11})(1 - f_{22}t_{22}) + f_{11}f_{22}t_{12}t_{21}} \begin{bmatrix} (1 - f_{22}t_{22}) & f_{11}t_{12} \\ f_{22}t_{21} & (1 - f_{11}t_{11}) \end{bmatrix}$$

$$l_{11} = \frac{1 - f_{22}t_{22}}{(1 - f_{11}t_{11})(1 - f_{22}t_{22}) + f_{11}f_{22}t_{12}t_{21}} \qquad (22)$$

$$l_{12} = \frac{f_{11}t_{12}}{(1 - f_{11}t_{11})(1 - f_{22}t_{22}) + f_{11}f_{22}t_{12}t_{21}} \qquad (23)$$

Since f_{11}, f_{22}, t_{11}, t_{12}.... all have real coefficients, the coefficients of l_{ij}, obtained by cross multiplication and summation of the above, are real numbers as well (statement no 1).

Since f_{11}, f_{22}, t_{11}, t_{12} .... all satisfy the condition: order of numerator $\leqslant$ order of denominator, so does their product. Considering terms of the type $f_{11}f_{22}$, or $f_{11}f_{22}t_{12}t_{21}$ as a whole, these terms will have a negative order (order numerator - order denominator $\leqslant 0$). Terms of the type $(1 - f_{22}t_{22})$ will have zero order (order numerator - order denominator = 0). Consequently, eq (22) shows that l_{11} satisfies the condition order of numerator = order of denominator. This is true for l_{ii} in general.

Similarly, eq (23) shows that l_{12} satisfies the condition: order of numerator $\leqslant$ order of denominator. This is true for all terms l_{ij} in general, where $i \neq j$.

To sum up, it is always true for <u>all</u> elements of L, that the order of numerator is $\leqslant$ order of denominator. Statement no 2 is thus shown to be valid. The above remarks are applicable to a matrix L of <u>any</u> order.

x^x x

The 3 statements will now be used to prove a realizability condition for C

$$C = (G + HT)L \qquad \text{where } L \triangleq (I - FT)^{-1}$$

For the (2x2) case:

$$C = \left[\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} + \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\right] \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$$

More explicitly,

$$c_{11} = (g_{11} + h_{11}t_{11} + h_{12}t_{21})l_{11} + (g_{12} + h_{11}t_{12} + h_{12}t_{22})l_{21} \qquad (24)$$

$$c_{12} = (g_{11} + h_{11}t_{11} + h_{12}t_{21})l_{12} + (g_{12} + h_{11}t_{12} + h_{12}t_{22})l_{22} \qquad (25)$$

A general expression for c_{ij} is:

$$c_{ij} = \sum_{k=1}^{n} \left(g_{ik} + \sum_{k=1}^{n} h_{ik}t_{kj} \right) l_{kj} \qquad (26)$$

In a (2x2) system, C_{ij} is the sum of 2 terms. In a (nxn) system, C_{ij} is the sum of n terms. Due to statement no 3, if each term is physically realizable in itself, then C_{ij} is realizable. But each of these terms has the form (see eq 24):

$$(g_{11} + h_{11}t_{11} + h_{12}t_{21} + \ldots.)l_{11} \tag{27}$$

which can be written as

$$\left(g_{11} + \sum_{k=1}^{n} h_{1k}t_{k1}\right) l_{11}$$

Each of the functions $h_{1k}t_{k1}$ is realizable, hence so is its sum (from statement no 3).

If g_{11} is also realizable, then so is $(g_{11} + \Sigma\, h_{1k}t_{k1})$.

But (27) also contains l_{11}. Thus another condition for realizability of (27) - ie: realizability of C_{ij} - is that l_{11}, too, be realizable. Statements no 1 and 2 show that l_{11} satisfies the first two conditions for realizability. The third condition is that l_{11} should have no forbidden poles. Equation (21) shows that poles of l_{11} are roots of the equation:

$$\left|L^{-1}\right| = 0$$

$$\text{or} \qquad \left|I - FT\right| = 0$$

(Other poles of l's may come from the expression of their respective numerators, but they are all poles of f's or of t's, ie: acceptable poles.)

The result for this general case of non-diagonal T matrix is summarized as follows. <u>If</u>:

a) <u>elements of G are physically realizable</u>

b) <u>roots of $|I - FT| = 0$ are on the LHP and are simple if on the imaginary axis,</u>

<u>then all indicated feedback compensation schemes can be realized.</u>

Note the generality of the above statement. In the particular case of diagonal T matrix, the roots of $|I - FT| = 0$ are the same as the zeros of (diagonal) elements of $(I - FT)$, ie: poles of elements of $(I - FT)^{-1}$, thus again yielding the results of sections III-1 to III-3.

Conclusion

Chapter Two of this report suggests a quick way to look into various feedback compensation schemes before making a judicious decision as to which scheme will best suit the designer's situation. Equations (11) (12) (13) are convenient for such purpose. For relatively small systems with 2, 3 variables, calculations are simple. For more complex systems, computational aids will be needed to avoid time consuming labor.

Chapter Three of the report develops a simple criterion to check whether or not a compensation scheme is realizable with physical components, using only the expression of G, the cascade open loop filter. The study begins with a non-interacting system and is later generalized to interacting systems as well. It turns out that if the elements of G are physically realizable and if, in addition, the determinant $|I - FT|$ has no roots on RHP, then feedback compensation schemes using any or all of the controllers C, F, H are realizable. This last restriction on $|I - FT|$ serves as a guide to the designer in the selection of F.

It is believed that the approach presented in this report leads to a systematic way of designing multivariable feedback control systems. Starting with the given plant P and the desired specification T, the designer obtains the expression of G, which, as shown in the report, is very convenient to work with, since the first step in obtaining a physically realizable feedback compensation scheme, is to try to make G itself realizable.

It is the authors' hope to report next on a method of obtaining a realizable G while satisfying other specifications on stability, transient response and interaction.

Appendix 1

Proof of statement no 3

If $f_1(s)$ and $f_2(s)$ are both physically realizable transfer functions, then $f_1(s) + f_2(s)$ is also a physically realizable transfer function

$$f_1(s) = \frac{a_n s^n + \dots + a_o}{b_m s^m + \dots + b_o} \qquad f_2(s) = \frac{\alpha_V s^V + \dots + \alpha_o}{\beta_w s^w + \dots + \beta_o}$$

where a, b, α, β are real coefficients

and $n \leqslant m$, $V \leqslant w$.

Now form the sum

$$f(s) = f_1(s) + f_2(s) =$$

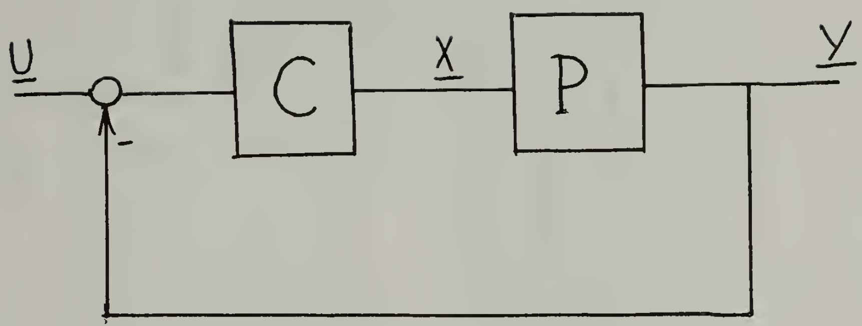
$$= \frac{\left(a_n s^n + \dots + a_o\right)\left(\beta_w s^w + \dots + \beta_o\right) + \left(\alpha_V s^V + \dots + \alpha_o\right)\left(b_m s^m + \dots + b_o\right)}{\left(b_m s^m + \dots + b_o\right)\left(\beta_w s^w + \dots + \beta_o\right)}$$

1 - coefficients of f(s) are real. (In denominator, products of real coefficients give real coefficients. In numerator the sum of products of real coefficients yield real coefficients also.)

2 - order of numerator of f(s) $\leqslant$ order of denominator.
Same proof is given in paragraph 2/ page 24.

3 - poles of f(s) are acceptable: poles of f(s) are merely those of $f_1(s)$ and those of $f_2(s)$ which are both acceptable.

Thus f(s) satisfies all 3 conditions and is realizable.



P = plant transfer function matrix

C = cascade controller transfer function matrix

U = system input column matrix

Y = system output column matrix

X = plant input column matrix

Figure 1 : A multivariable control system

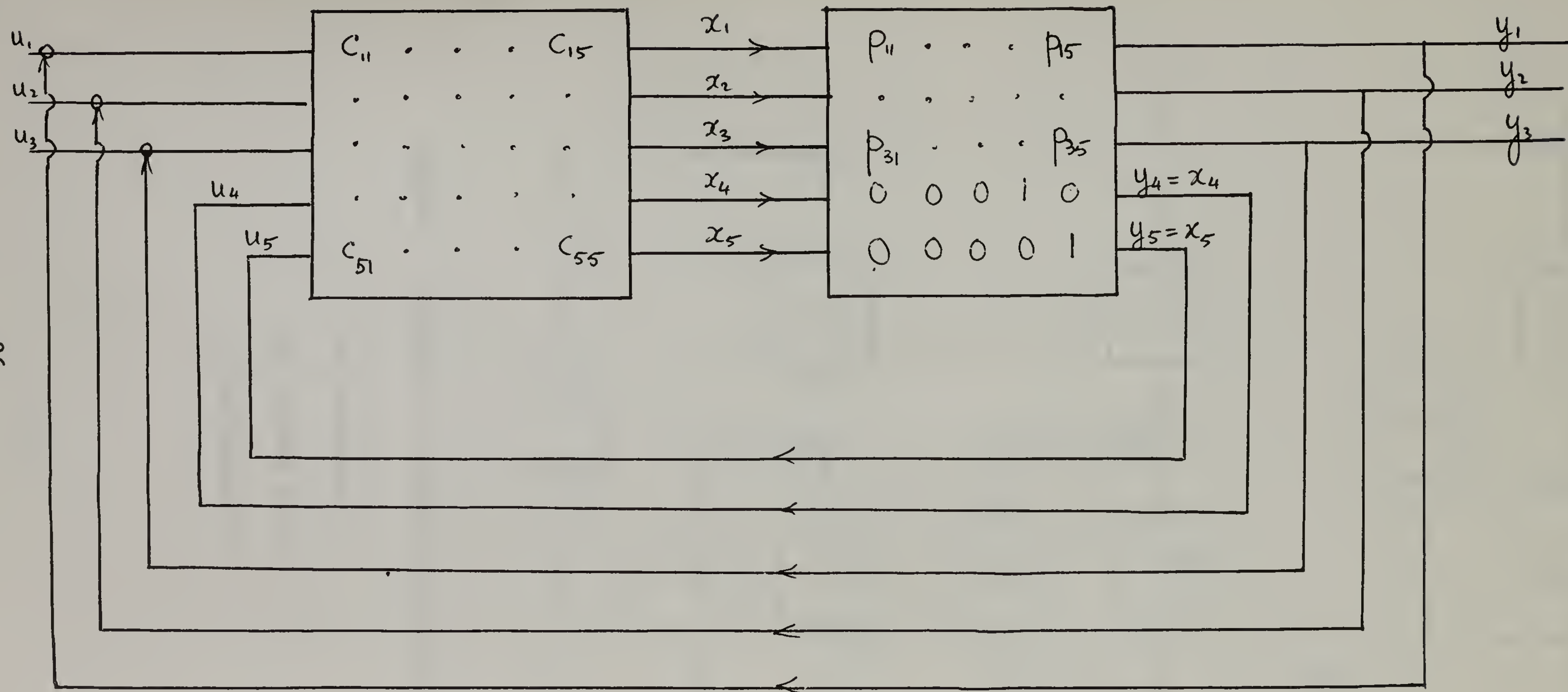


Figure 2 : Multivariable control system
with P modified to a square matrix

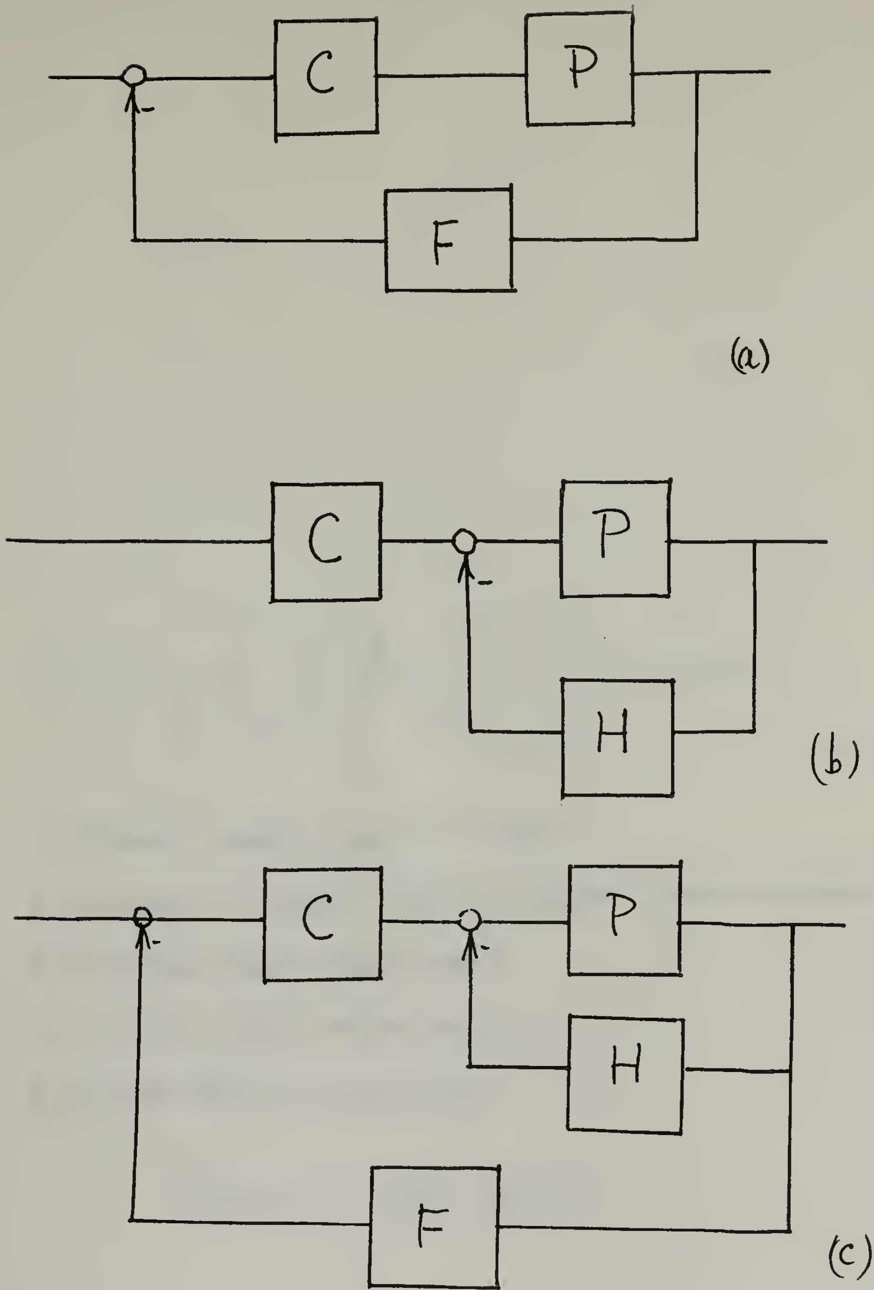
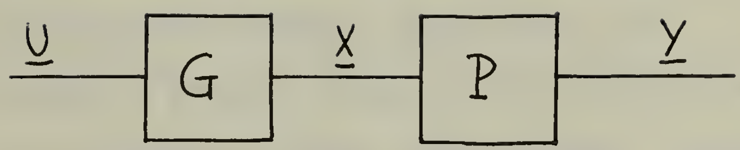


Figure 3 : Possible feedback structures

(a) scheme no 1

(b) scheme no 2

(c) scheme no 3



P = plant transfer function matrix

G = open-loop cascade filter transfer function matrix

U = system input column matrix

Y = system output column matrix

X = plant input column matrix

Figure 4 : Filter design

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